

Constructions in Ramsey theory

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Abstract

We provide several constructions for problems in Ramsey theory. First, we prove a superexponential lower bound for the classical 4-uniform Ramsey number $r_4(5, n)$, and the same for the iterated $(k-4)$ -fold logarithm of the k -uniform version $r_k(k+1, n)$. This is the first improvement of the original exponential lower bound for $r_4(5, n)$ implicit in work of Erdős and Hajnal from 1972 and also improves the current best known bounds for larger k due to the authors. Second, we prove an upper bound for the hypergraph Erdős-Rogers function $f_{k+1, k+2}^k(N)$ that is an iterated $(k-13)$ -fold logarithm in N . This improves the previous upper bounds that were only logarithmic and addresses a question of Dudek and the first author that was reiterated by Conlon, Fox and Sudakov. Third, we generalize the results of Erdős and Hajnal about the 3-uniform Ramsey number of K_4 minus an edge versus a clique to k -uniform hypergraphs. Finally, we observe that Ramsey graphs can be transformed to tournaments with no large transitive subtournament, and this gives explicit constructions of superpolynomial size for the Erdős-Moser problem about tournaments with no large transitive subtournament.

1 Introduction

A k -uniform hypergraph H (k -graph for short) with vertex set V is a collection of k -element subsets of V . We write K_n^k for the complete k -uniform hypergraph on an n -element vertex set. Given k -graphs F, G , the *Ramsey number* $r(F, G)$ is the minimum N such that every red/blue coloring of the edges of K_N^k results in a monochromatic red copy of F or a monochromatic blue copy of G .

In this paper, we study several problems in hypergraph Ramsey theory. We describe each problem in detail in its relevant section. Here we provide a brief summary. In Section 2, we give new lower bounds on the classical Ramsey number $r(K_{k+1}^k, K_n^k)$, improving the previous best known bounds obtained by the authors [25]. In particular, we give the first superexponential lower bound for $r(K_5^4, K_n^4)$ since the problem was first explicitly stated by Erdős and Hajnal [19] in 1972. In Section 3, we establish a new upper bound for the hypergraph Erdős-Rogers function $f_{k+1, k+2}^k(N)$ that is an iterated logarithm function in N . More precisely, we construct k -graphs on N vertices, with no copy of K_{k+2}^k yet every set of n vertices contains a copy of K_{k+1}^k where n is the $(k-13)$ -fold iterated logarithm of N . This addresses questions posed by Dudek and the first author [13] as well

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as by Conlon, Fox, and Sudakov [11] and significantly improves the previous best known bound in [13] of $n = O((\log N)^{1/(k-1)})$. In Section 4 we study the Ramsey numbers for k -half-graphs versus cliques, generalizing the results of Erdős and Hajnal [19] about the 3-uniform Ramsey number of K_4 minus an edge versus a clique. The upper bound is a straightforward extension of the method in [19], while the constructions are new. Finally in Section 5 we show how to transform Ramsey graphs to tournaments with no large transitive subtournament. While this transformation is very easy, and appears in the literature in other contexts, we observe that it gives explicit constructions of superpolynomial size for the Erdős-Moser problem about tournaments with no large transitive subtournament. Specifically, using the current best known explicit Ramsey graphs of Cohen [7] and Chattopadhyay and Zuckerman [6], we obtain explicit constructions of tournaments of size $2^{2^{(\log n)^c}}$ for some $c > 0$ and n large that contain no transitive subtournament of size n .

All logarithms are base 2 unless otherwise stated. For the sake of clarity of presentation, we systematically omit floor and ceiling signs whenever they are not crucial.

2 A new lower bound for $r_k(k+1, n)$

In order to avoid the excessive use of superscripts, we use the simpler notation $r(K_s^k, K_n^k) = r_k(s, n)$. Estimating the Ramsey number $r_k(s, n)$ is a classical problem in extremal combinatorics and has been extensively studied [20, 21, 23]. Here we study the *off-diagonal* Ramsey number, that is, $r_k(s, n)$ with k, s fixed and n tending to infinity. It is known that for fixed $s \geq k+1$, $r_2(s, n)$ grows polynomial in n [1, 3, 4] and $r_3(s, n)$ grows exponential in a power of n [10]. In 1972, Erdős and Hajnal [19] raised the question of determining the correct tower growth rate for $r_k(s, n)$. We define *tower function* $\text{twr}_k(x)$ by

$$\text{twr}_1(x) = x \quad \text{and} \quad \text{twr}_{i+1}(x) = 2^{\text{twr}_i(x)}.$$

By applying the Erdős-Hajnal stepping up lemma in the off-diagonal setting (see [24]), it follows that $r_k(s, n) \geq \text{twr}_{k-1}(\Omega(n))$, for $k \geq 4$ and for all $s \geq 2^{k-1} - k + 3$. However they conjectured the following.

Conjecture 2.1. (Erdős-Hajnal [19]) *For $s \geq k+1 \geq 5$ fixed, $r_k(s, n) \geq \text{twr}_{k-1}(\Omega(n))$.*

In [9], Conlon, Fox, and Sudakov modified the Erdős-Hajnal stepping-up lemma to show that Conjecture 2.1 holds for all $s \geq \lceil 5k/2 \rceil - 3$. Recently the authors nearly proved the conjecture by establishing the following.

Theorem 2.2 ([25]). *There is a positive constant $c > 0$ such that the following holds. For $k \geq 4$ and $n > 3k$, we have*

1. $r_k(k+3, n) \geq \text{twr}_{k-1}(cn)$,
2. $r_k(k+2, n) \geq \text{twr}_{k-1}(c \log^2 n)$,
3. $r_k(k+1, n) \geq \text{twr}_{k-2}(cn^2)$.

Implicit in work of Erdős and Hajnal [19] is the bound $r_4(5, n) > 2^{cn}$ for some absolute positive constant c . While the authors [25] recently improved this to 2^{cn^2} above, there has been no superexponential lower bound given for this basic problem. Here we provide such a lower bound.

Theorem 2.3. *There is an absolute constant $c > 0$ such that*

$$r_4(5, n) > 2^{n^{c \log \log n}}$$

and more generally for $k > 4$,

$$r_k(k+1, n) > \text{twr}_{k-2}(n^{c \log \log n}).$$

One of the building blocks we will use in our construction is the following lower bound of Conlon, Fox, and Sudakov [10]: there is an absolute positive constant $c > 0$ such that

$$r_3(4, t) > 2^{ct \log t}. \quad (1)$$

Our lower bound for $r_4(5, n)$ is proved via the following theorem.

Theorem 2.4. *For n sufficiently large, we have*

$$r_4(5, n) > 2^{r_3(4, \lfloor \log n/2 \rfloor) - 1}.$$

Proof. The idea is to apply a variant of the Erdős-Hajnal stepping up lemma (see [24]). Set $t = \lfloor \frac{\log n}{2} \rfloor$. Let ϕ be a red/blue coloring of the edges of the complete 3-uniform hypergraph on the vertex set $\{0, 1, \dots, r_3(4, t) - 2\}$ without a red K_4^3 and without a blue K_t^3 . We use ϕ to define a red/blue coloring χ of the edges of the complete 4-uniform hypergraph K_N^4 on the vertex set $V = \{0, 1, \dots, N - 1\}$ with $N = 2^{r_3(4, t) - 1}$, as follows.

For any $a \in V$, write $a = \sum_{i=0}^{r_3(4, t) - 2} a(i) 2^i$ with $a(i) \in \{0, 1\}$ for each i . For $a \neq b$, let $\delta(a, b)$ denote the largest i for which $a(i) \neq b(i)$. Notice that we have the following stepping-up properties (again see [24])

Property A: For every triple $a < b < c$, $\delta(a, b) \neq \delta(b, c)$.

Property B: For $a_1 < \dots < a_r$, $\delta(a_1, a_r) = \max_{1 \leq j \leq r-1} \delta(a_j, a_{j+1})$.

Given any 4-tuple $a_1 < \dots < a_4$ of V , consider the integers $\delta_i = \delta(a_i, a_{i+1})$, $1 \leq i \leq 3$. Say that $\delta_1, \delta_2, \delta_3$ forms a monotone sequence if $\delta_1 < \delta_2 < \delta_3$ or $\delta_1 > \delta_2 > \delta_3$. Now, define χ as follows:

$$\chi(a_1, a_2, a_3, a_4) = \begin{cases} \phi(\delta_1, \delta_2, \delta_3) & \text{if } \delta_1, \delta_2, \delta_3 \text{ is monotone} \\ \text{blue} & \text{if } \delta_1, \delta_2, \delta_3 \text{ is not monotone} \end{cases}$$

Hence we have the following property which can be easily verified using Properties A and B (see [24]).

Property C: For $a_1 < \dots < a_r$, set $\delta_j = \delta(a_j, a_{j+1})$ and suppose that $\delta_1, \dots, \delta_{r-1}$ form a monotone sequence. If χ colors every 4-tuple in $\{a_1, \dots, a_r\}$ red (blue), then ϕ colors every triple in $\{\delta_1, \dots, \delta_{r-1}\}$ red (blue).

For sake of contradiction, suppose that the coloring χ produces a red K_5^4 on vertices $a_1 < \dots < a_5$, and let $\delta_i = \delta(a_i, a_{i+1})$, $1 \leq i \leq 4$. Then $\delta_1, \dots, \delta_4$ form a monotone sequence and, by Property C, ϕ colors every triple in $\{\delta_1, \dots, \delta_4\}$ red which is a contradiction. Therefore, there is no red K_5^4 in coloring χ .

Next we show that there is no blue K_n^4 in coloring χ . Our argument is reminiscent of the standard argument for the bound $r_2(n, n) < 4^n$, though it must be adapted to this setting. For sake of contradiction, suppose we have vertices $a_1, \dots, a_n \in V$ such that $a_1 < \dots < a_n$ and χ colors every 4-tuple in the set $\{a_1, \dots, a_n\}$ blue. Let $\delta_i = \delta(a_i, a_{i+1})$ for $1 \leq i \leq n-1$. We greedily construct a set $D_h = \{\delta_{i_1}, \dots, \delta_{i_h}\} \subset \{\delta_1, \dots, \delta_{n-1}\}$ and a set $S_h \subset \{a_1, \dots, a_n\}$ such that the following holds.

1. We have $\delta_{i_1} > \dots > \delta_{i_h}$.
2. For each $\delta_{i_j} = \delta(a_{i_j}, a_{i_j+1}) \in D_h = \{\delta_{i_1}, \dots, \delta_{i_h}\}$, consider the set of vertices

$$A = \{a_{i_j+1}, a_{i_j+1+1}, \dots, a_{i_h}, a_{i_h+1}\} \cup S_h.$$

Then either every element in A is greater than a_{i_j} or every element in A is less than a_{i_j+1} . In the former case we will label δ_{i_j} *white*, in the latter case we label it *black*.

3. The indices of the vertices in S_h are consecutive, that is, $S_h = \{a_r, a_{r+1}, \dots, a_{s-1}, a_s\}$ for $1 \leq r < s \leq n$.

We start with the $D_0 = \emptyset$ and $S_0 = \{a_1, \dots, a_n\}$. Having obtained $D_h = \{\delta_{i_1}, \dots, \delta_{i_h}\}$ and $S_h = \{a_r, \dots, a_s\}$, $1 \leq r < s \leq n$, we construct D_{h+1} and S_{h+1} as follows. Let $\delta_{i_{h+1}} = \delta(a_\ell, a_{\ell+1})$ be the unique largest element in $\{\delta_r, \delta_{r+1}, \dots, \delta_{s-1}\}$, and set $D_{h+1} = D_h \cup \delta_{i_{h+1}}$. The uniqueness of $\delta_{i_{h+1}}$ follows from Properties A and B. If $|\{a_r, a_{r+1}, \dots, a_\ell\}| \geq |S_h|/2$, then we set $S_{h+1} = \{a_r, a_{r+1}, \dots, a_\ell\}$. Otherwise by the pigeonhole principle, we have $|\{a_{\ell+1}, a_{\ell+2}, \dots, a_s\}| \geq |S_h|/2$ and we set $S_{h+1} = \{a_{\ell+1}, a_{\ell+2}, \dots, a_s\}$.

Since $|S_0| = n$, $t = \lfloor \frac{\log n}{2} \rfloor$ and $|S_{h+1}| \geq |S_h|/2$ for $h \geq 0$, we can construct $D_{2t} = \{\delta_{i_1}, \dots, \delta_{i_{2t}}\}$ with the desired properties. By the pigeonhole principle, at least t elements in D_{2t} have the same label, say *white*. The other case will follow by a symmetric argument. We remove all black labeled elements in D_{2t} , and let $\{\delta_{j_1}, \dots, \delta_{j_t}\}$ be the resulting set. Now consider the vertices $a_{j_1}, a_{j_2}, \dots, a_{j_t} \in V$. By construction and by Property B, we have $a_{j_1} < a_{j_2} < \dots < a_{j_t}$ and $\delta(a_{j_1}, a_{j_2}) = \delta_{i_{j_1}}, \delta(a_{j_2}, a_{j_3}) = \delta_{i_{j_2}}, \dots, \delta(a_{j_t}, a_{j_{t+1}}) = \delta_{i_{j_t}}$. Therefore we have a monotone sequence

$$\delta(a_{j_1}, a_{j_2}) > \delta(a_{j_2}, a_{j_3}) > \dots > \delta(a_{j_t}, a_{j_{t+1}}).$$

By Property C, ϕ colors every triple from this set blue which is a contradiction. Therefore there is no red K_5^4 and no blue K_n^4 in coloring χ . \square

Applying the lower bound in (1), we obtain that

$$r_4(5, n) \geq 2^{r_3(4, \lfloor \log n/2 \rfloor) - 1} > 2^{2^c \log n \log \log n} = 2^{n^c \log \log n}$$

for some absolute positive constant c and this establishes the first part of Theorem 2.3.

We next prove Theorem 2.3 for $k \geq 5$. Independently, Conlon, Fox and Sudakov [8] gave a different proof of Theorem 2.2 part 1. Their approach was to begin with a known 4-uniform construction that yields $r_4(7, n) > 2^{2^{cn}}$ and then use a variant of the stepping up lemma to give tower-type lower bounds for larger k . Unfortunately, this variant of the stepping up lemma does not work if one begins instead with a lower bound for $r_4(5, n)$ which is our case. However, a further variant of the approach does work, and this is what we do below.

Lemma 2.5. *For $k \geq 5$ and n sufficiently large, we have*

$$r_k(k+1, n) \geq 2^{r_{k-1}(k, \lfloor n/6 \rfloor) - 1}.$$

Proof. Again we apply a variant of the stepping-up lemma. Let ϕ be a red/blue coloring of the edges of the complete $(k-1)$ -uniform hypergraph on the vertex set $\{0, 1, \dots, r_{k-1}(k, \lfloor n/6 \rfloor) - 2\}$ without a red K_k^{k-1} and without a blue $K_{\lfloor n/6 \rfloor}^{k-1}$. We use ϕ to define a red/blue coloring χ of the edges of the complete k -uniform hypergraph K_N^k on the vertex set $V = \{0, 1, \dots, N-1\}$ with $N = 2^{r_{k-1}(k, \lfloor n/6 \rfloor) - 1}$, as follows.

Just as above, for any $a \in V$, write $a = \sum_{i=0}^{r_{k-1}(k, \lfloor n/6 \rfloor) - 2} a(i)2^i$ with $a(i) \in \{0, 1\}$ for each i . For $a \neq b$, let $\delta(a, b)$ denote the largest i for which $a(i) \neq b(i)$. Hence Properties A and B hold.

Given any k -tuple $a_1 < a_2 < \dots < a_k$ of V , consider the integers $\delta_i = \delta(a_i, a_{i+1})$, $1 \leq i \leq k-1$. We say that δ_i is a *local minimum* if $\delta_{i-1} > \delta_i < \delta_{i+1}$, a *local maximum* if $\delta_{i-1} < \delta_i > \delta_{i+1}$, and a *local extremum* if it is either a local minimum or a local maximum. We say that δ_i is *locally monotone* if $\delta_{i-1} < \delta_i < \delta_{i+1}$ or $\delta_{i-1} > \delta_i > \delta_{i+1}$. Since $\delta_{i-1} \neq \delta_i$ for every i , every nonmonotone sequence $\delta_1, \dots, \delta_{k-1}$ has a local extremum. If $\delta_1, \dots, \delta_{k-1}$ form a monotone sequence, then let $\chi(a_1, a_2, \dots, a_k) = \phi(\delta_1, \delta_2, \dots, \delta_{k-1})$. Otherwise if $\delta_1, \dots, \delta_{k-1}$ is not monotone, then let $\chi(a_1, a_2, \dots, a_k)$ be red if and only if δ_2 is a local maximum and δ_3 is a local minimum. Hence the following generalization of Property C holds.

Property D: For $a_1 < \dots < a_r$, set $\delta_j = \delta(a_j, a_{j+1})$ and suppose that $\delta_1, \dots, \delta_{r-1}$ form a monotone sequence. If χ colors every k -tuple in $\{a_1, \dots, a_r\}$ red (blue), then ϕ colors every $(k-1)$ -tuple in $\{\delta_1, \dots, \delta_{r-1}\}$ red (blue).

For sake of contradiction, suppose that the coloring χ produces a red K_{k+1}^k on vertices $a_1 < \dots < a_{k+1}$, and let $\delta_i = \delta(a_i, a_{i+1})$, $1 \leq i \leq k$. We have two cases.

Case 1. Suppose $\delta_1, \dots, \delta_{k-1}$ is monotone. Then if $\delta_2, \dots, \delta_k$ is also a monotone sequence, ϕ colors every $(k-1)$ -tuple in $\{\delta_1, \dots, \delta_k\}$ red by Property D, which is a contradiction. Otherwise, δ_{k-1} is the only local extremum and $\chi(a_2, \dots, a_{k+1})$ is blue, which is again a contradiction.

Case 2. Suppose $\delta_1, \dots, \delta_{k-1}$ is not monotone. Then we know that δ_2 is a local maximum and δ_3 is a local minimum. However this implies that $\chi(a_2, \dots, a_{k+1})$ is blue, which is a contradiction. Hence there is no red K_{k+1}^k in coloring χ .

Next we show that there is no blue K_n^k in coloring χ . For sake of contradiction, suppose we have vertices $a_1, \dots, a_n \in V$ such that $a_1 < \dots < a_n$ and χ colors every k -tuple blue, and let

$\delta_i = \delta(a_i, a_{i+1})$ for $1 \leq i \leq n-1$. By Property D, there is no integer r such that $\delta_r, \delta_{r+1}, \dots, \delta_{r+\lfloor n/6 \rfloor}$ is monotone, since this implies that ϕ colors every $(k-1)$ -tuple in the set $\{\delta_r, \delta_{r+1}, \dots, \delta_{r+\lfloor n/6 \rfloor}\}$ blue which is a contradiction. Therefore the sequence $\delta_1, \dots, \delta_{n-1}$ contains at least four local extremums. Let δ_{j_1} be the first local maximum, and let δ_{j_2} be the next local extremum, which must be a local minimum. Recall that $\delta_{j_1} = \delta(a_{j_1}, a_{j_1+1})$ and $\delta_{j_2} = \delta(a_{j_2}, a_{j_2+1})$. Consider the k vertices

$$a_{j_1-1}, a_{j_1}, a_{j_2}, a_{j_2+1}, a_{j_2+2}, \dots, a_{j_2+k-3}$$

and the sequence

$$\delta(a_{j_1-1}, a_{j_1}), \delta(a_{j_1}, a_{j_2}), \delta(a_{j_2}, a_{j_2+1}), \dots, \delta(a_{j_2+k-4}, a_{j_2+k-3}).$$

By Property B we have $\delta(a_{j_1}, a_{j_2}) = \delta_{j_1}$, and therefore $\delta(a_{j_1}, a_{j_2})$ is a local maximum and $\delta(a_{j_2}, a_{j_2+1})$ is a local minimum. Therefore $\chi(a_{j_1-1}, a_{j_1}, a_{j_2}, a_{j_2+1}, \dots, a_{j_2+k-3})$ is red and we have our contradiction. Hence there is no blue K_n^k in coloring χ . \square

By combining Theorem 2.4 with Lemma 2.5, we establish Theorem 2.3.

3 The Erdős-Rogers function for hypergraphs

An s -independent set in a k -graph H is a vertex subset that contains no copy of K_s^k . So if $s = k$, then it is just an independent set. Let $\alpha_s(H)$ denote the size of the largest s -independent set in H .

Definition 3.1. For $k \leq s < t < N$, the Erdős-Rogers function $f_{s,t}^k(N)$ is the minimum of $\alpha_s(H)$ taken over all K_t^k -free k -graphs H of order N .

To prove the lower bound $f_{s,t}^k(N) \geq n$ one must show that every K_t^k -free k -graph of order N contains an s -independent set with n vertices. On the other hand, to prove the upper bound $f_{s,t}^k(N) < n$, one must construct a K_t^k -free k -graph H of order N with $\alpha_s(H) < n$.

The problem of determining $f_{s,t}^k(n)$ extends that of finding Ramsey numbers. Formally,

$$r_k(s, n) = \min\{N : f_{k,s}^k(N) \geq n\}.$$

For $k = 2$ the above function was first considered by Erdős and Rogers [22] only for $t = s + 1$, which might be viewed as the most restrictive case. Since then the function has been studied by several researchers culminating in the work of Wolfowitz [29] and Dudek, Retter and Rödl [14] who proved the upper bound that follows (the lower bound is due to Dudek and the first author [13]): for every $s \geq 3$ there are positive constants c_1 and $c_2(s)$ such that

$$c_1 \left(\frac{N \log N}{\log \log N} \right)^{1/2} < f_{s,s+1}^2(N) < c_2(\log N)^{4s^2} N^{1/2}.$$

The problem of estimating the Erdős-Rogers function for $k > 2$ appears to be much harder. Let us denote

$$g(k, N) = f_{k+1,k+2}^k(N)$$

so that the above result (for $s = 3$) becomes $g(2, N) = N^{1/2+o(1)}$. Dudek and the first author [13] proved that $(\log N)^{1/4+o(1)} < g(3, N) < O(\log N)$ and more generally that there are positive constants c_1 and c_2 with

$$c_1(\log_{(k-2)} N)^{1/4} < g(k, N) < c_2(\log N)^{1/(k-2)} \quad (2)$$

where $\log_{(i)}$ is the log function iterated i times. The exponent $1/4$ was improved to $1/3$ by Conlon, Fox, Sudakov [11]. Both sets of authors asked whether the upper bound could be improved (presumably to an iterated log function). Here we prove this where the number of iterations is $k - O(1)$. It remains an open problem to determine the correct number of iterations (which may well be $k - 2$).

Theorem 3.2. *Fix $k \geq 14$. Then $g(k, N) < O(\log_{(k-13)} N)$.*

Proof. We may assume that $k \geq 15$ else we use the upper bound in (2). We will proceed by induction on k so assume that the result holds for $k - 1$ and we want to prove it for $k \geq 15$. We will show that

$$g(k, 2^N) < k \cdot g(k - 1, N),$$

and this recurrence clearly implies the theorem. Indeed, it easily implies the upper bound

$$g(k, N) < 2^k k! \log_{(k-13)} N$$

by induction on k , as $g(k + 1, N)$ is at most

$$\begin{aligned} g(k + 1, 2^{\lceil \log N \rceil}) &< (k + 1)g(k, \lceil \log N \rceil) \\ &< 2^k (k + 1)! \log_{(k-13)} \lceil \log N \rceil \\ &\leq 2^{k+1} (k + 1)! \log_{(k-12)} N. \end{aligned}$$

Our strategy is to apply a variant of the stepping-up lemma. Let us begin with a K_{k+1}^{k-1} -free $(k - 1)$ -graph H' on N vertices for which $\alpha_k(H') = g(k - 1, N)$. Note that this exists by definition of $g(k - 1, N)$. We will use H' to produce a K_{k+2}^k -free k -graph H on 2^N vertices with $\alpha_{k+1}(H) < k\alpha_k(H') = kg(k - 1, N)$.

Let $V(H') = \{0, 1, \dots, N - 1\}$ and $V(H) = \{0, 1, \dots, 2^N - 1\}$. For any $a \in V(H)$, write $a = \sum_{i=0}^{N-1} a(i)2^i$ with $a(i) \in \{0, 1\}$ for each i . For $a \neq b$, let $\delta(a, b)$ denote the largest i for which $a(i) \neq b(i)$. Therefore Properties A and B in the previous section hold.

Given any set of s vertices $a_1 < a_2 < \dots < a_s$ of $V(H)$, consider the integers $\delta_i = \delta(a_i, a_{i+1})$, $1 \leq i \leq s - 1$. For $e = (a_1, \dots, a_s)$, let $m(e)$ denote the number of local extrema in the sequence $\delta_1, \dots, \delta_{s-1}$. In the case $s = k$, we define the edges of H as follows. If $\delta_1, \dots, \delta_{k-1}$ form a monotone sequence, then let $(a_1, a_2, \dots, a_k) \in E(H)$ if and only if $(\delta_1, \delta_2, \dots, \delta_{k-1}) \in E(H')$. Otherwise if $\delta_1, \dots, \delta_{k-1}$ is not monotone, then $(a_1, a_2, \dots, a_k) \in E(H)$ if and only if $m(e) \in \{k - 4, k - 3\}$. In other words, given that $\delta_1, \dots, \delta_{k-1}$ is not monotone, $(a_1, a_2, \dots, a_k) \in E(H)$ if and only if $\delta_1, \dots, \delta_{k-1}$ has at most one locally monotone element. Note that we have the following variant of Property D.

Property E: For $a_1 < \dots < a_r$, set $\delta_j = \delta(a_j, a_{j+1})$ and suppose that $\delta_1, \dots, \delta_{r-1}$ form a monotone sequence. If every k -tuple in $\{a_1, \dots, a_r\}$ is in $E(H)$ (in $\overline{E}(H)$), then every $(k-1)$ -tuple in $\{\delta_1, \dots, \delta_{r-1}\}$ is in $E(H')$ (in $\overline{E}(H')$).

We are to show that H contains no $(k+2)$ -clique and $\alpha_{k+1}(H) < k\alpha_k(H')$. First let us establish the following lemma.

Lemma 3.3. *Given $e = (a_1, \dots, a_7)$ with $a_1 < \dots < a_7$, let $\delta_i = \delta(a_i, a_{i+1})$ for $1 \leq i \leq 6$. If $m(e) = 4$, then there is an a_i such that $2 \leq i \leq 6$ and $m(e - a_i) = 2$.*

Proof. Suppose first that δ_2 is a local minimum, so $\delta_1 > \delta_2 < \delta_3 > \dots$. If $\delta_1 > \delta_3$, then we delete a_3 and by property B we have $\delta(a_2, a_4) = \delta_3$. Hence $\delta_1 > \delta(a_2, a_4) > \delta_4$ and $m(e - a_2) = 2$. So we may assume that $\delta_1 < \delta_3$. Now we repeat the argument we just did with δ_3, δ_4 and δ_5 to conclude that $\delta_3 < \delta_5$. In this case we delete a_5 and by Property B we have $\delta_2 < \delta_3 < \delta(a_4, a_6)$. Therefore we have $m(e - a_5) = 2$.

If δ_2 is a local maximum with $\delta_4 > \delta_2$, then we delete a_4 and by Property B we have $\delta_1 < \delta_2 < \delta_4 = \delta(a_3, a_5)$. Hence $m(e - a_4) = 2$. If $\delta_4 < \delta_2$, then we apply the argument in the previous paragraph starting with a_2 . \square

For sake of contradiction, suppose there are $k+2$ vertices $a_1 < \dots < a_{k+2}$ that induces a K_{k+2}^k in H . Define $\delta_i = \delta(a_i, a_{i+1})$ for all $1 \leq i \leq k+1$. Given the sequence $\delta_1, \delta_2, \dots, \delta_{k+1}$, let us consider the number of locally monotone elements in $D = \{\delta_2, \dots, \delta_k\}$.

Case 1. Suppose every element in D is locally monotone. Then $\delta_1, \dots, \delta_{k+1}$ form a monotone sequence. By Property E, every $(k-1)$ -tuple in the set $\{\delta_1, \dots, \delta_{k+1}\}$ is an edge in H' which is a contradiction since H' is K_{k+1}^{k-1} -free.

Case 2. Suppose there is at least one local extremum $\delta_\ell \in D$ and at least two elements $\delta_i, \delta_j \in D$ that are locally monotone. Then any k -tuple $e \subset \{a_1, \dots, a_{k+2}\}$ that includes the vertices

$$a_{i-1}, a_i, a_{i+1}, a_{i+2}, a_{j-1}, a_j, a_{j+1}, a_{j+2}, a_{\ell-1}, a_\ell, a_{\ell+1}, a_{\ell+2}$$

satisfies $1 \leq m(e) < k-4$. Therefore e is not an edge in H and we have a contradiction.

Case 3. Suppose there is exactly one element $\delta_i \in D$ that is locally monotone (and therefore at least one local extremum). Since $k \geq 15$, either $|\{a_1, \dots, a_{i-1}\}| \geq 7$ or $|\{a_{i+2}, \dots, a_{k+2}\}| \geq 7$. Let us only consider the former case, the latter being symmetric. By Lemma 3.3, there is an element $a_j \in \{a_2, \dots, a_6\} \subset \{a_1, \dots, a_{i-1}\}$ such that for $e' = (a_1, \dots, a_7)$, $m(e' - a_j) = 2$. Then any k -tuple $e \subset \{a_1, \dots, a_{k+2}\} \setminus \{a_j\}$ that includes vertices

$$\{a_t : 1 \leq t \leq 7, t \neq j\} \cup \{a_{i-1}, a_i, a_{i+1}, a_{i+2}\}$$

satisfies $1 \leq m(e) < k-4$. Hence e is not an edge in H and we have a contradiction.

Case 4. Suppose every element in D is a local extremum. We then apply Lemma 3.3 to the set $A = \{a_1, \dots, a_7\}$ and $B = \{a_8, \dots, a_{14}\}$ to obtain vertices $a_i \in A$ and $a_j \in B$ such that $m(\{a_1, \dots, a_7\} \setminus \{a_i\}) = 2$ and $m(\{a_8, \dots, a_{14}\} \setminus \{a_j\}) = 2$. In particular, this implies that for $e = \{a_1, \dots, a_{k+2}\} \setminus \{a_i, a_j\}$, the corresponding sequence of δ 's has at least two locally monotone

elements. Since clearly e has at least one local extremum, we obtain $1 \leq m(e) < k - 4$. Hence $e \notin E(H)$ and we have a contradiction.

Therefore we have shown that H is K_{k+2}^k -free.

Our final task is to show that $\alpha_{k+1}(H) < k\alpha_k(H')$. Set $n = kt$ where $t = \alpha_k(H')$. Let us assume for contradiction that there are vertices $a_1 < \dots < a_n$ that induces a $(k+1)$ -independent set in H . Let $\delta_i = \delta(a_i, a_{i+1})$ for $1 \leq i \leq n-1$. If the sequence $\delta_1, \dots, \delta_{n-1}$ contains less than k local extrema, then there is a j such that $\delta_j, \dots, \delta_{j+t}$ is monotone. Since $t = \alpha_k(H')$, the $t+1$ vertices $\{\delta_j, \dots, \delta_{j+t}\}$ contains a copy of K_k^{k-1} in H' . Say this copy is given by $\delta_{j_1}, \dots, \delta_{j_k}$. Then by Property E, the vertices $a_{j_1} < \dots < a_{j_k} < a_{j_k+1}$ induces a copy of K_{k+1}^k which contradicts our assumption that $\{a_1, \dots, a_n\}$ is a $(k+1)$ -independent set in H .

We may therefore assume that the sequence $\delta_1, \dots, \delta_{n-1}$ contains at least k local extrema. Now we make the following claim.

Claim 3.4. *There is a set of $k+1$ vertices $a_1^*, \dots, a_{k+1}^* \in \{a_1, \dots, a_n\}$ such that for $\delta_i^* = \delta(a_i^*, a_{i+1}^*)$, the sequence $\delta_1^*, \dots, \delta_k^*$ has $k-2$ local extrema.*

Proof. Let $\delta_{i_1}, \dots, \delta_{i_k}$ be the first k extrema in the sequence $\delta_1, \dots, \delta_{n-1}$.

Case 1. Suppose δ_{i_1} is a local minimum. If k is odd, then consider the $k+1$ distinct vertices

$$e = a_{i_1}, a_{i_1+1}, a_{i_3}, a_{i_3+1}, a_{i_5}, a_{i_5+1}, \dots, a_{i_k}, a_{i_k+1}.$$

Note that the pairs $(a_{i_1}, a_{i_1+1}), (a_{i_3}, a_{i_3+1}), (a_{i_5}, a_{i_5+1}), \dots$ correspond to local minimums. By Property B, $\delta(a_{i_1+1}, a_{i_3}) = \delta_{i_2}$, $\delta(a_{i_3+1}, a_{i_5}) = \delta_{i_4}, \dots$. Since $\delta_{i_2}, \delta_{i_4}, \delta_{i_6}, \dots$ were local maximums in the sequence $\delta_1, \dots, \delta_{n-1}$, we have

$$\delta(a_{i_1}, a_{i_1+1}) < \delta(a_{i_1+1}, a_{i_3}) > \delta(a_{i_3}, a_{i_3+1}) < \delta(a_{i_3+1}, a_{i_5}) > \dots.$$

Hence the vertices in e satisfies the claim. If k is even, then by the same argument as above, the $k+1$ vertices

$$a_1, a_{i_1}, a_{i_1+1}, a_{i_3}, a_{i_3+1}, a_{i_5}, a_{i_5+1}, \dots, a_{i_{k-1}}, a_{i_{k-1}+1}$$

satisfies the claim.

Case 2. Suppose δ_{i_1} is a local maximum. If k is odd, then the arguments above implies that the set of $k+1$ vertices

$$a_1, a_2, a_{i_2}, a_{i_2+1}, a_{i_4}, a_{i_4+1}, \dots, a_{i_{k-1}}, a_{i_{k-1}+1}$$

satisfies the claim. Likewise, if k is even, the set of $k+1$ vertices

$$a_1, a_{i_2}, a_{i_2+1}, a_{i_4}, a_{i_4+1}, \dots, a_{i_k}, a_{i_k+1}$$

satisfies the claim. \square

By Claim 3.4, we obtain $k + 1$ vertices $h = (a_1^*, \dots, a_{k+1}^*)$ along with $\delta_1^*, \dots, \delta_k^*$ with the desired properties. Consider the k -tuple $e = h - a_i^*$. If $i = 1$ or $k + 1$, then it is easy to see that $m(e) = k - 3$, which implies $e \in E(H)$. For $i = 2$, δ_3^* is the only possible locally monotone element in the sequence $\delta(a_1^*, a_3^*), \delta_3^*, \dots, \delta_k^*$. Therefore $m(e - a_i) \geq k - 4$ and $e \in E(H)$. A symmetric argument for the case $i = k$ implies that $e \in E(H)$. Therefore we can assume $3 \leq i \leq k - 1$. By Property B, we have $\delta(a_{i-1}^*, a_{i+1}^*) = \max\{\delta_{i-1}^*, \delta_i^*\}$. Let us consider the two cases.

Case 1. Suppose $\delta(a_{i-1}^*, a_{i+1}^*) = \delta_{i-1}^*$. If $\delta_{i+1}^* > \delta_{i-1}^*$, then δ_{i-1}^* is the only element in the sequence $\delta_1^*, \dots, \delta_{i-1}^*, \delta_{i+1}^*, \dots, \delta_k^*$ that is locally monotone. Hence $m(e) = k - 4$ and $e \in E(H)$. If $\delta_{i+1}^* < \delta_{i-1}^*$, then δ_{i+1}^* is the only possible element in the sequence $\delta_1^*, \dots, \delta_{i-1}^*, \delta_{i+1}^*, \dots, \delta_k^*$ that is locally monotone. More precisely, if $i = k - 1$ then $m(e) = k - 3$, and if $3 \leq i < k - 1$ then $m(e) = k - 4$. Hence $m(e) \geq k - 4$ and therefore $e \in E(H)$.

Case 2. Suppose $\delta(a_{i-1}^*, a_{i+1}^*) = \delta_i^*$. If $\delta_{i-2}^* > \delta_i^*$, then δ_i^* is the only element in the sequence $\delta_1^*, \dots, \delta_{i-2}^*, \delta_i^*, \dots, \delta_k^*$ that is locally monotone. Hence $m(e) = k - 4$ and $e \in E(H)$. If $\delta_{i-2}^* < \delta_i^*$, then δ_{i-2}^* is the only possible element in the sequence $\delta_1^*, \dots, \delta_{i-2}^*, \delta_i^*, \dots, \delta_k^*$ that is locally monotone. More precisely, if $i = 3$ then $m(e) = k - 3$, and if $3 < i \leq k - 1$ then $m(e) = k - 4$. Hence $m(e) \geq k - 4$ and $e \in E(H)$.

Therefore every k -tuple $e = h - a_i$ is an edge in H , and the $k + 1$ vertices h induces a K_{k+1}^k in H . This is a contradiction and we have completed the proof. \square

4 Ramsey numbers for k -half-graphs versus cliques

Let $K_4^3 \setminus e$ denote the 3-uniform hypergraph on four vertices, obtained by removing one edge from K_4^3 . A simple argument of Erdős and Hajnal [19] implies $r(K_4^3 \setminus e, K_n^3) < (n!)^2$. On the other hand, they also gave a construction that shows $r(K_4^3 \setminus e, K_n^3) > 2^{cn}$ for some constant $c > 0$. Improving either of these bounds is a very interesting open problem, as $K_4^3 \setminus e$ is, in some sense, the smallest 3-uniform hypergraph whose Ramsey number with a clique is at least exponential.

A k -half-graph, denote by B^k , is a k -uniform hypergraph on $2k - 2$ vertices, whose vertex set is of the form $S \cup T$, where $|S| = |T| = k - 1$, and whose edges are all k -subsets that contain S , and one k -subset that contains T . The hypergraph B^k can be viewed as a generalization of $K_4^3 \setminus e$ as $B^3 = K_4^3 \setminus e$.

The goal of this section is to obtain upper and lower bounds for $r(B^k, K_n^k)$ that parallel the known state of affairs for $K_4^3 \setminus e$. We begin by presenting a straightforward generalization of the argument of Erdős and Hajnal to establish an upper bound for Ramsey numbers for k -half-graphs versus cliques. Again for simplicity we write $r(B^k, K_n^k) = r_k(B, n)$.

Theorem 4.1. *For $k \geq 4$, we have $r_k(B, n) \leq (n!)^{k-1}$.*

First, let us recall an old lemma due to Spencer.

Lemma 4.2 ([26]). *Let $H = (V, E)$ be a k -uniform hypergraph on N vertices. If $|E(H)| > N/k$, then there exists a subset $S \subset V(H)$ such that S is an independent set and*

$$|S| \geq \left(1 - \frac{1}{k}\right) N \left(\frac{N}{k|E(H)|}\right)^{\frac{1}{k-1}}.$$

Proof of Theorem 4.1. We proceed by induction on n . The base case $n = k$ is trivial. Let $n > k$ and assume the statement holds for $n' < n$. Let $k \geq 4$ and let χ be a red/blue coloring on the edges of K_N^k , where $N = (n!)^{k-1}$. Let E_R denote the set of red edges in K_N^k .

Case 1: Suppose $|E_R| < \frac{(1-\frac{1}{k})^{k-1} N^k}{kn^{k-1}}$. Then by Lemma 4.2, K_N^k contains a blue clique of size n .

Case 2: Suppose $|E_R| \geq \frac{(1-\frac{1}{k})^{k-1} N^k}{kn^{k-1}}$. Then by averaging, there is a $(k-1)$ -element subset $S \subset V$ such that $N(S) = \{v \in V : S \cup \{v\} \in E_R\}$ satisfies

$$|N(S)| \geq \frac{(1 - \frac{1}{k})^{k-1} N^k}{n^{k-1} \binom{n}{k-1}} \geq ((n-1)!)^{k-1}.$$

The last inequality follows from the fact that $k \geq 4$. Fix a vertex $u \in S$. If $\{u\} \cup T \in E_R$ for some $T \subset N(S)$ such that $|T| = k-1$, then $S \cup T$ forms a red B^k and we are done. Therefore we can assume otherwise. By the induction hypothesis, $N(S)$ contains a red copy of B^k , or a blue copy of K_{n-1}^k . We are done in the former case, and the latter case, we can form a blue K_n^k by adding the vertex u . \square

We now move to our main new contribution, which are constructions which show that $r_k(B, n)$ is at least exponential in n .

Theorem 4.3. *For fixed $k \geq 3$, we have $r_k(B, n) > 2^{\Omega(n)}$.*

Proof. Surprisingly, we require different arguments for k even and k odd.

The case when k is odd. Assume k is odd, and set $N = 2^{cn}$ where $c = c(k)$ will be determined later. Then let T be a random tournament on the vertex set $[N]$, that is, for $i, j \in [N]$, independently, either $(i, j) \in E$ or $(j, i) \in E$, where each of the two choices is equally likely. Then let $\chi : \binom{[N]}{k} \rightarrow \{\text{red}, \text{blue}\}$ be a red/blue coloring on the k -subsets of $[N]$, where $\chi(v_1, \dots, v_k) = \text{red}$ if v_1, \dots, v_k induces a *regular* tournament, that is, the indegree of every vertex is $(k-1)/2$ (and hence the outdegree of every vertex is $(k-1)/2$). Otherwise we color it blue. We note that since k is odd, a regular tournament on k vertices is possible by the fact that K_k has an Eulerian circuit, and then by directing the edges according to the circuit we obtain a regular tournament.

Notice that the coloring χ does not contain a red B^k . Indeed, let $S, T \subset [N]$ such that $|S| = |T| = k-1$, $S \cap T = \emptyset$, and every k -tuple of the form $S \cup \{v\}$ is red, for all $v \in T$. Then for any $u \in S$, all edges in the set $u \times T$ must have the same direction, either all emanating out of u or all directed towards u . Therefore it is impossible for $u \cup T$ to have color red, for any choice $u \in S$.

Next we estimate the expected number of monochromatic blue copies of K_n^k in χ . For a given k -tuple $v_1, \dots, v_k \in [N]$, the probability that $\chi(v_1, \dots, v_k) = \text{blue}$ is clearly at most $1 - 1/2^{\binom{k}{2}}$. Let $T = \{v_1, \dots, v_n\}$ be a set of t vertices in $[n]$, where $v_1 < \dots < v_n$. Let S be a partial Steiner $(n, k, 2)$ -system with vertex set T , that is, S is a k -uniform hypergraph such that each 2-element set of vertices is contained in at most one edge in S . Moreover, S satisfies $|S| = c_k n^2$. It is known that such a system exists. Then the probability that every k -tuple in T has color blue is at most the probability that every k -tuple in S is blue. Since the edges in S are independent, that is no two edges have more than one vertex in common, the probability that T is a monochromatic blue clique is at most $\left(1 - 1/2^{\binom{k}{2}}\right)^{|S|} \leq \left(1 - 1/2^{\binom{k}{2}}\right)^{c_k n^2}$. Therefore the expected number of monochromatic blue copies of K_n^k in χ is at most

$$\binom{N}{n} \left(1 - 1/2^{\binom{k}{2}}\right)^{c_k n^2} < 1,$$

for an appropriate choice for $c = c(k)$. Hence, there is a coloring χ with no red B^k and no blue K_n^k . Therefore

$$r_k(B, n) > 2^{cn}.$$

The case when k is even. Assume k is even and set $N = 2^{ct}$ where $c = c(k)$ will be determined later. Consider the coloring $\phi : \binom{[N]}{2} \rightarrow \{1, \dots, k-1\}$, where each edge has probability $1/(k-1)$ of being a particular color independent of all other edges (pairs). Using ϕ , we define the coloring $\chi : \binom{[N]}{k} \rightarrow \{\text{red}, \text{blue}\}$, where the k -tuple (v_1, \dots, v_k) is red if ϕ is a proper edge-coloring on all pairs among $\{v_1, \dots, v_k\}$, that is, each of the $k-1$ colors appears as a perfect matching. Otherwise we color it blue.

Notice that the coloring χ does not contain a red B^k . Indeed let $S, T \subset [N]$ such that $|S| = |T| = k-1$ and $S \cap T = \emptyset$. If every k -tuple of the form $S \cup \{v\}$ is red, for all $v \in T$, then all $k-1$ colors from ϕ appear among the edges (pairs) in the set $S \times \{v\}$. Hence for any vertex $u \in S$, χ could not have colored $u \cup T$ red since it is impossible to have *any* of the $k-1$ colors to appear as a perfect matching in $u \cup T$.

For a given k -tuple $v_1, \dots, v_k \in [N]$, the probability that $\chi(v_1, \dots, v_k) = \text{blue}$ is at most $1 - (1/(k-1))^{\binom{k}{2}}$. By the same argument as above, the expected number of monochromatic blue copies of K_n^k with respect to χ is less than 1 for an appropriate choice of $c = c(k)$. Hence, there is a coloring χ with no red B^k and no blue K_n^k . Therefore

$$r_k(B, n) > 2^{cn}$$

and the proof is complete. □

5 Transitive tournaments

A *tournament* on a set V is an orientation $T = (V, E)$ of the edges of the complete graph with vertex set V , that is, for $u, v \in V$ we have either $(u, v) \in E$ or $(v, u) \in E$, but not both. A tournament is transitive if for every $u, v, w \in V$ such that $(u, v), (v, w) \in E$ we have $(u, w) \in E$.

Let $T(n)$ denote the smallest integer N such that every tournament on N vertices contains a transitive subtournament on n vertices. Erdős and Moser [18] and Stearns [27] proved that

$$2^{(n-1)/2} < T(n) < 2^n.$$

Starting with a tournament $T = ([N], E)$ we can construct a red/blue coloring of the complete graph K on $[N]$ where for $i < j$ the pair $\{i, j\}$ is red iff $(i, j) \in E$. Then the vertex set of a monochromatic clique in K is a transitive subtournament in T . This immediately yields the well-known inequality

$$T(n) \leq r(n, n).$$

In the other direction, we can use the known results for $r(n, n)$ and $T(n)$ to obtain

$$r(n, n) < 4^n = 2^{((4n+1)-1)/2} < T(4n+1). \quad (3)$$

The lower bound for $T(n)$ in (3) is obtained by the probabilistic method, and it appears that there are no known explicit constructions that have superpolynomial size. Indeed, the inequalities in (3) do not say anything about how to produce a construction for $T(n)$ from one for $r(n, n)$. Here we show how to convert a Ramsey graph to such a construction via the following result (which gives an alternative proof of the weaker bound $r(n, n) \leq T(n^2)$).

We remark that this result is not new, and appears, for example in a paper of Alon, Pach and Solymosi [2] in relation to the Erdős-Hajnal problem. Our main point here is that it can be used to give *explicit* constructions of tournaments with no large transitive subtournament.

Theorem 5.1. *Let χ be a red/blue coloring of the edges of the complete graph K_N with vertex set $[N]$ that contains no monochromatic clique of size n . Let $T = ([N], E)$ be the tournament where for $i < j$ we have $(i, j) \in E$ if $\chi(i, j)$ is red, and $(j, i) \in E$ if $\chi(i, j)$ is blue. Then T contains no transitive subtournament of size n^2 .*

Proof. Suppose for contradiction there is a subset $U \subset [N]$ such that $|U| = n^2$ that induces a transitive subtournament. Let $T' = (U, E)$. Notice that for $u, v, w \in U$, if $(u, v), (v, w) \in E$ and $\chi(u, v) = \chi(v, w)$ is blue (red), then $(u, w) \in E$ and $\chi(u, w)$ is blue (red).

A well known result in graph theory states that every transitive tournament contains a unique Hamiltonian path (see [28]). We now order the vertices in U with respect to the Hamiltonian path, that is, $U = \{u_1, \dots, u_{n^2}\}$, where $(u_i, u_{i+1}) \in E$ for all $1 \leq i \leq n^2 - 1$. Since T' is transitive, we have $(u_i, u_j) \in E$ for all $i < j$. By the Erdős-Szekeres Theorem [23], there are n indices $j_1 < j_2 < \dots < j_n$ such that $\chi(u_{j_i}, u_{j_{i+1}})$ is blue (red) for all $1 \leq i \leq n - 1$. By the observation above and since T' is transitive, $\{u_{j_1}, \dots, u_{j_n}\}$ induces a monochromatic clique in K , contradiction. \square

Theorem 5.1 easily gives an efficient construction of a tournament with no large transitive subtournament from a Ramsey graph. Indeed, we start with a red/blue coloring of K_N with vertex set

$[N]$ and no monochromatic clique of size n , where $N = r(n, n) - 1$. Then we form the tournament T as in Theorem 5.1. We can now combine this with any of the known explicit constructions of Ramsey graphs (e.g., due to Boaz, Rao, Shaltiel, Wigderson [5], or the recent improvements due to Cohen [7], or Chattopadhyay and Zuckerman [6]). Using the best known constructions, we obtain the following.

Corollary 5.2. *There is a $c > 0$ and an n_0 such that for $n > n_0$, there is an explicit construction of a tournament on $N > 2^{2^{(\log n)^c}}$ vertices, with no transitive subtournament of size n .*

Note that the constructions in [5, 7, 6] are explicit according to the formal definition (they can be constructed in polynomial time), but very hard to describe. Easier (very explicit) constructions due to Frankl and Wilson [16] can also be used above with the slightly weaker result $N > 2^{c \log^2 n}$.

Acknowledgments. We thank Noga Alon, David Conlon and Pedro Vieira informing us about the references [2, 6, 7].

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